39. Numerically estimate $\lim_{x\to 0^+} x^{\sec x}$. Try to numerically estimate $\lim_{x\to 0^-} x^{\sec x}$. If your computer has difficulty evaluating the function for negative x's, explain why.

40. Explain what is wrong with the following logic (note from exercise 39 that the answer is accidentally correct): since 0 to any power is 0, $\lim_{x\to 0} x^{\sec x} = \lim_{x\to 0} 0^{\sec x} = 0$.

41. Give an example of a function f such that $\lim_{x\to 0} f(x)$ exists but $f(0)$ does not exist. Give an example of a function g such that $g(0)$ exists but $\lim_{x\to 0} g(x)$ does not exist.

42. Give an example of a function f such that $\lim_{x\to 0} f(x)$ exists and $f(0)$ exists, but $\lim_{x\to 0} f(x) \neq f(0)$.

43. In the text, we described $\lim_{x\to a} f(x) = L$ as meaning "as x gets closer and closer to a, $f(x)$ is getting closer and closer to L." As x gets closer and closer to 0, it is true that x^2 gets closer and closer to -0.01, but it is certainly not true that $\lim_{x\to 0} x^2 = -0.01$. Try to modify the description of limit to make it clear that $\lim_{x\to 0} x^2 \neq -0.01$. We explore a very precise definition of limit in section 1.6.

44. In Figure 1.13, the final position of the knuckleball at time $t = 0.68$ is shown as a function of the rotation rate ω. The batter must decide at time $t = 0.4$ whether to swing at the pitch. At $t = 0.4$, the left/right position of the ball is given by $h(\omega) = \dfrac{1}{\omega} - \dfrac{5}{8\omega^2}\sin(1.6\omega)$. Graph $h(\omega)$ and compare to Figure 1.13. Conjecture the limit of $h(\omega)$ as $\omega \to 0$. For $\omega = 0$, is there any difference in ball position between what the batter sees at $t = 0.4$ and what he tries to hit at $t = 0.68$?

45. A parking lot charges \$2 for each hour or portion of an hour, with a maximum charge of \$12 for all day. If $f(t)$ equals the total parking bill for t hours, sketch a graph of $y = f(t)$ for $0 \le t \le 24$. Determine the limits $\lim_{t\to 3.5} f(t)$ and $\lim_{t\to 4} f(t)$, if they exist.

46. For the parking lot in exercise 45, determine all values of a with $0 \le a \le 24$ such that $\lim_{t\to a} f(t)$ does not exist. Briefly discuss the effect this has on your parking strategy (e.g., are there times where you would be in a hurry to move your car or times where it doesn't matter whether you move your car?).

EXPLORATORY EXERCISES

1. In a situation similar to that of example 2.6, the left/right position of a knuckleball pitch in baseball can be modeled by $P = \dfrac{5}{8\omega^2}(1 - \cos 4\omega t)$, where t is time measured in seconds $(0 \le t \le 0.68)$ and ω is the rotation rate of the ball measured in radians per second. In example 2.6, we chose a specific t-value and evaluated the limit as $\omega \to 0$. While this gives us some information about which rotation rates produce hard-to-hit pitches, a clearer picture emerges if we look at P over its entire domain. Set $\omega = 10$ and graph the resulting function $\dfrac{1}{160}(1 - \cos 40t)$ for $0 \le t \le 0.68$. Imagine looking at a pitcher from above and try to visualize a baseball starting at the pitcher's hand at $t = 0$ and finally reaching the batter, at $t = 0.68$. Repeat this with $\omega = 5$, $\omega = 1$, $\omega = 0.1$ and whatever values of ω you think would be interesting. Which values of ω produce hard-to-hit pitches?

2. In this exercise, the results you get will depend on the accuracy of your computer or calculator. Work this exercise and compare your results with your classmates' results. We will investigate $\lim_{x\to 0} \dfrac{\cos x - 1}{x^2}$. Start with the calculations presented in the table (your results may vary):

x	$f(x)$
0.1	$-0.499583\ldots$
0.01	$-0.49999583\ldots$
0.001	$-0.4999999583\ldots$

Describe as precisely as possible the pattern shown here. What would you predict for $f(0.0001)$? $f(0.00001)$? Does your computer or calculator give you this answer? If you continue trying powers of 0.1 (0.000001, 0.0000001 etc.) you should eventually be given a displayed result of -0.5. Do you think this is exactly correct or has the answer just been rounded off? Why is rounding off inescapable? It turns out that -0.5 is the exact value for the limit, so the round-off here is somewhat helpful. However, if you keep evaluating the function at smaller and smaller values of x, you will eventually see a reported function value of 0. This round-off error is not so benign; we discuss this error in section 1.7. For now, evaluate $\cos x$ at the current value of x and try to explain where the 0 came from.

1.3 COMPUTATION OF LIMITS

Now that you have an idea of what a limit is, we need to develop some means of calculating limits of simple functions. In this section, we present some basic rules for dealing with common limit problems. We begin with two simple limits.

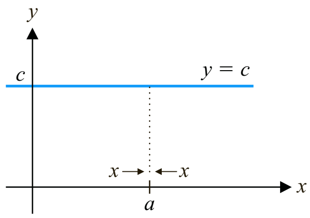

FIGURE 1.14
$\lim_{x \to a} c = c$

For any constant c and any real number a,

$$\lim_{x \to a} c = c. \tag{3.1}$$

In other words, the limit of a constant is that constant. This certainly comes as no surprise, since the function $f(x) = c$ does not depend on x and so, stays the same as $x \to a$ (see Figure 1.14). Another simple limit is the following.

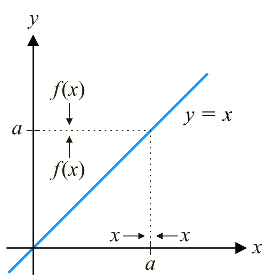

FIGURE 1.15
$\lim_{x \to a} x = a$

For any real number a,

$$\lim_{x \to a} x = a. \tag{3.2}$$

Again, this is not a surprise, since as $x \to a$, x will approach a (see Figure 1.15). Be sure that you are comfortable enough with the limit notation to recognize how obvious the limits in (3.1) and (3.2) are. As simple as they are, we use them repeatedly in finding more complex limits. We also need the basic rules contained in Theorem 3.1.

THEOREM 3.1

Suppose that $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ both exist and let c be any constant. The following then apply:

(i) $\lim_{x \to a} [c \cdot f(x)] = c \cdot \lim_{x \to a} f(x)$,

(ii) $\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$,

(iii) $\lim_{x \to a} [f(x) \cdot g(x)] = \left[\lim_{x \to a} f(x)\right]\left[\lim_{x \to a} g(x)\right]$ and

(iv) $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} \left(\text{if } \lim_{x \to a} g(x) \neq 0\right)$.

The proof of Theorem 3.1 is found in Appendix A and requires the formal definition of limit discussed in section 1.6. You should think of these rules as sensible results that you would certainly expect to be true, given your intuitive understanding of what a limit is. Read them in plain English. For instance, part (ii) says that the limit of a sum (or a difference) equals the sum (or difference) of the limits, *provided the limits exist*. Think of this as follows. If as x approaches a, $f(x)$ approaches L and $g(x)$ approaches M, then $f(x) + g(x)$ should approach $L + M$.

Observe that by applying part (iii) of Theorem 3.1 with $g(x) = f(x)$, we get that, whenever $\lim_{x \to a} f(x)$ exists,

$$\begin{aligned} \lim_{x \to a} [f(x)]^2 &= \lim_{x \to a} [f(x) \cdot f(x)] \\ &= \left[\lim_{x \to a} f(x)\right]\left[\lim_{x \to a} f(x)\right] = \left[\lim_{x \to a} f(x)\right]^2. \end{aligned}$$

Likewise, for any positive integer n, we can apply part (iii) of Theorem 3.1 repeatedly, to yield

$$\lim_{x \to a} [f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n \tag{3.3}$$

(see exercises 67 and 68).

Notice that taking $f(x) = x$ in (3.3) gives us that for any integer $n > 0$ and any real number a,

$$\lim_{x \to a} x^n = a^n. \tag{3.4}$$

That is, to compute the limit of any positive power of x, you simply substitute in the value of x being approached.

EXAMPLE 3.1 Finding the Limit of a Polynomial

Apply the rules of limits to evaluate $\lim_{x \to 2} (3x^2 - 5x + 4)$.

Solution We have

$$\begin{aligned}
\lim_{x \to 2} (3x^2 - 5x + 4) &= \lim_{x \to 2} (3x^2) - \lim_{x \to 2} (5x) + \lim_{x \to 2} 4 && \text{By Theorem 3.1 (ii).} \\
&= 3 \lim_{x \to 2} x^2 - 5 \lim_{x \to 2} x + 4 && \text{By Theorem 3.1 (i).} \\
&= 3 \cdot (2)^2 - 5 \cdot 2 + 4 = 6. && \text{By (3.4).} \quad \blacksquare
\end{aligned}$$

EXAMPLE 3.2 Finding the Limit of a Rational Function

Apply the rules of limits to evaluate $\lim_{x \to 3} \dfrac{x^3 - 5x + 4}{x^2 - 2}$.

Solution We get

$$\begin{aligned}
\lim_{x \to 3} \frac{x^3 - 5x + 4}{x^2 - 2} &= \frac{\lim_{x \to 3} (x^3 - 5x + 4)}{\lim_{x \to 3} (x^2 - 2)} && \text{By Theorem 3.1 (iv).} \\
&= \frac{\lim_{x \to 3} x^3 - 5 \lim_{x \to 3} x + \lim_{x \to 3} 4}{\lim_{x \to 3} x^2 - \lim_{x \to 3} 2} && \text{By Theorem 3.1 (i) and (ii).} \\
&= \frac{3^3 - 5 \cdot 3 + 4}{3^2 - 2} = \frac{16}{7}. && \text{By (3.4).} \quad \blacksquare
\end{aligned}$$

You may have noticed that in examples 3.1 and 3.2, we simply ended up substituting the value for x, after taking many intermediate steps. In example 3.3, it's not quite so simple.

EXAMPLE 3.3 Finding a Limit by Factoring

Evaluate $\lim_{x \to 1} \dfrac{x^2 - 1}{1 - x}$.

Solution Notice right away that

$$\lim_{x \to 1} \frac{x^2 - 1}{1 - x} \neq \frac{\lim_{x \to 1} (x^2 - 1)}{\lim_{x \to 1} (1 - x)},$$

since the limit in the denominator is zero. (Recall that the limit of a quotient is the quotient of the limits *only* when both limits exist *and* the limit in the denominator is *not*

zero.) We can resolve this problem by observing that

$$\lim_{x\to 1}\frac{x^2-1}{1-x}=\lim_{x\to 1}\frac{(x-1)(x+1)}{-(x-1)}$$ Factoring the numerator and factoring -1 from denominator.

$$=\lim_{x\to 1}\frac{(x+1)}{-1}=-2,$$ Simplifying and substituting $x=1$.

where the cancellation of the factors of $(x-1)$ is valid because in the limit as $x\to 1$, x is close to 1, but $x\neq 1$, so that $x-1\neq 0$. ■

In Theorem 3.2, we show that the limit of a polynomial is simply the value of the polynomial at that point; that is, to find the limit of a polynomial, we simply substitute in the value that x is approaching.

THEOREM 3.2

For any polynomial $p(x)$ and any real number a,

$$\lim_{x\to a}p(x)=p(a).$$

PROOF

Suppose that $p(x)$ is a polynomial of degree $n\geq 0$,

$$p(x)=c_nx^n+c_{n-1}x^{n-1}+\cdots+c_1x+c_0.$$

Then, from Theorem 3.1 and (3.4),

$$\begin{aligned}\lim_{x\to a}p(x)&=\lim_{x\to a}(c_nx^n+c_{n-1}x^{n-1}+\cdots+c_1x+c_0)\\&=c_n\lim_{x\to a}x^n+c_{n-1}\lim_{x\to a}x^{n-1}+\cdots+c_1\lim_{x\to a}x+\lim_{x\to a}c_0\\&=c_na^n+c_{n-1}a^{n-1}+\cdots+c_1a+c_0=p(a).\ ■\end{aligned}$$

Evaluating the limit of a polynomial is now easy. Many other limits are evaluated just as easily.

THEOREM 3.3

Suppose that $\lim_{x\to a}f(x)=L$ and n is any positive integer. Then,

$$\lim_{x\to a}\sqrt[n]{f(x)}=\sqrt[n]{\lim_{x\to a}f(x)}=\sqrt[n]{L},$$

where for n even, we assume that $L>0$.

The proof of Theorem 3.3 is given in Appendix A. Notice that this result says that we may (under the conditions outlined in the hypotheses) bring limits "inside" nth roots. We can then use our existing rules for computing the limit inside.

EXAMPLE 3.4 Evaluating the Limit of an nth Root of a Polynomial

Evaluate $\lim_{x\to 2}\sqrt[5]{3x^2-2x}$.

Solution By Theorems 3.2 and 3.3, we have

$$\lim_{x\to 2}\sqrt[5]{3x^2-2x}=\sqrt[5]{\lim_{x\to 2}(3x^2-2x)}=\sqrt[5]{8}.$$ ■

REMARK 3.1

In general, in any case where the limits of both the numerator and the denominator are 0, you should try to algebraically simplify the expression, to get a cancellation, as we do in examples 3.3 and 3.5.

EXAMPLE 3.5 Finding a Limit by Rationalizing

Evaluate $\lim_{x\to 0} \dfrac{\sqrt{x+2}-\sqrt{2}}{x}$.

Solution First, notice that both the numerator ($\sqrt{x+2}-\sqrt{2}$) and the denominator (x) approach 0 as x approaches 0. Unlike example 3.3, we can't factor the numerator. However, we can rationalize the numerator, as follows:

$$\frac{\sqrt{x+2}-\sqrt{2}}{x} = \frac{(\sqrt{x+2}-\sqrt{2})(\sqrt{x+2}+\sqrt{2})}{x(\sqrt{x+2}+\sqrt{2})} = \frac{x+2-2}{x(\sqrt{x+2}+\sqrt{2})}$$
$$= \frac{x}{x(\sqrt{x+2}+\sqrt{2})} = \frac{1}{\sqrt{x+2}+\sqrt{2}},$$

where the last equality holds if $x \neq 0$ (which is the case in the limit as $x \to 0$). So, we have

$$\lim_{x\to 0} \frac{\sqrt{x+2}-\sqrt{2}}{x} = \lim_{x\to 0} \frac{1}{\sqrt{x+2}+\sqrt{2}} = \frac{1}{\sqrt{2}+\sqrt{2}} = \frac{1}{2\sqrt{2}}.$$

So that we are not restricted to discussing only the algebraic functions (i.e., those that can be constructed by using addition, subtraction, multiplication, division, exponentiation and by taking nth roots), we state the following result now, without proof.

THEOREM 3.4

For any real number a, we have

(i) $\lim_{x\to a} \sin x = \sin a$,

(ii) $\lim_{x\to a} \cos x = \cos a$,

(iii) $\lim_{x\to a} e^x = e^a$ and

(iv) $\lim_{x\to a} \ln x = \ln a$, for $a > 0$.

(v) $\lim_{x\to a} \sin^{-1} x = \sin^{-1} a$, for $-1 < a < 1$,

(vi) $\lim_{x\to a} \cos^{-1} x = \cos^{-1} a$, for $-1 < a < 1$,

(vii) $\lim_{x\to a} \tan^{-1} x = \tan^{-1} a$, for $-\infty < a < \infty$ and

(viii) if p is a polynomial and $\lim_{x\to p(a)} f(x) = L$, then $\lim_{x\to a} f(p(x)) = L$.

Notice that Theorem 3.4 says that limits of the sine, cosine, exponential, natural logarithm, inverse sine, inverse cosine and inverse tangent functions are found simply by substitution. A more thorough discussion of functions with this property (called *continuity*) is found in section 1.4.

EXAMPLE 3.6 Evaluating a Limit of an Inverse Trigonometric Function

Evaluate $\lim_{x\to 0} \sin^{-1}\left(\dfrac{x+1}{2}\right)$.

Solution By Theorem 3.4, we have

$$\lim_{x\to 0} \sin^{-1}\left(\frac{x+1}{2}\right) = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}.$$

So much for limits that we can compute using elementary rules. Many limits can be found only by using more careful analysis, often by an indirect approach. For instance, consider the following problem.

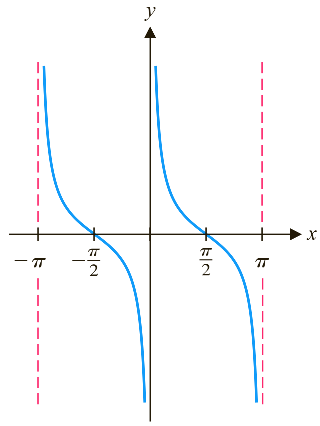

FIGURE 1.16
$y = \cot x$

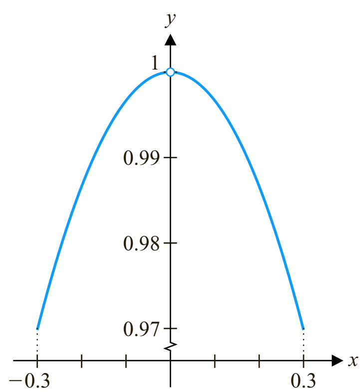

FIGURE 1.17
$y = x \cot x$

x	$x \cot x$
± 0.1	0.9967
± 0.01	0.999967
± 0.001	0.99999967
± 0.0001	0.9999999967
± 0.00001	0.999999999967

EXAMPLE 3.7 A Limit of a Product That Is Not the Product of the Limits

Evaluate $\lim_{x\to 0}(x \cot x)$.

Solution Your first reaction might be to say that this is a limit of a product and so, must be the product of the limits:

$$\lim_{x\to 0}(x \cot x) = \left(\lim_{x\to 0} x\right)\left(\lim_{x\to 0} \cot x\right) \quad \text{This is incorrect!}$$

$$= 0 \cdot ? = 0, \tag{3.5}$$

where we've written a "?" since you probably don't know what to do with $\lim_{x\to 0} \cot x$. Since the first limit is 0, do we really need to worry about the second limit? The problem here is that we are attempting to apply the result of Theorem 3.1 in a case where the hypotheses are not satisfied. Specifically, Theorem 3.1 says that the limit of a product is the product of the respective limits *when all of the limits exist.* The graph in Figure 1.16 suggests that $\lim_{x\to 0} \cot x$ does not exist. You should compute some function values, as well, to convince yourself that this is in fact the case. So, equation (3.5) does not hold and we're back to square one. Since none of our rules seem to apply here, the most reasonable step is to draw a graph (see Figure 1.17) and compute some function values. Based on these, we conjecture that

$$\lim_{x\to 0}(x \cot x) = 1,$$

which is definitely not 0, as you might have initially suspected. You can also think about this limit as follows:

$$\lim_{x\to 0}(x \cot x) = \lim_{x\to 0}\left(x \frac{\cos x}{\sin x}\right) = \lim_{x\to 0}\left(\frac{x}{\sin x}\cos x\right)$$

$$= \left(\lim_{x\to 0} \frac{x}{\sin x}\right)\left(\lim_{x\to 0} \cos x\right)$$

$$= \frac{\lim_{x\to 0} \cos x}{\lim_{x\to 0} \dfrac{\sin x}{x}} = \frac{1}{1} = 1,$$

since $\lim_{x\to 0} \cos x = 1$ and where we have used the conjecture we made in example 2.4 that $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$. (We verify this last conjecture in section 2.6, using the Squeeze Theorem, which follows.) ■

At this point, we introduce a tool that will help us determine a number of important limits.

THEOREM 3.5 (Squeeze Theorem)

Suppose that

$$f(x) \le g(x) \le h(x)$$

for all x in some interval (c, d), except possibly at the point $a \in (c, d)$ and that

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L,$$

for some number L. Then, it follows that

$$\lim_{x\to a} g(x) = L, \text{ also.}$$

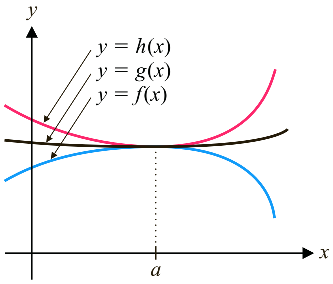

FIGURE 1.18
The Squeeze Theorem

The proof of Theorem 3.5 is given in Appendix A, since it depends on the precise definition of limit found in section 1.6. However, if you refer to Figure 1.18, you should clearly see that if $g(x)$ lies between $f(x)$ and $h(x)$, except possibly at a itself and both $f(x)$ and $h(x)$ have the same limit as $x \to a$, then $g(x)$ gets *squeezed* between $f(x)$ and $h(x)$ and therefore should also have a limit of L. The challenge in using the Squeeze Theorem is in finding appropriate functions f and h that bound a given function g from below and above, respectively, and that have the same limit as $x \to a$.

REMARK 3.2

The Squeeze Theorem also applies to one-sided limits.

EXAMPLE 3.8 Using the Squeeze Theorem to Verify the Value of a Limit

Determine the value of $\displaystyle\lim_{x\to 0}\left[x^2\cos\left(\frac{1}{x}\right)\right]$.

Solution Your first reaction might be to observe that this is a limit of a product and so, might be the product of the limits:

$$\lim_{x\to 0}\left[x^2\cos\left(\frac{1}{x}\right)\right] \stackrel{?}{=} \left(\lim_{x\to 0} x^2\right)\left[\lim_{x\to 0}\cos\left(\frac{1}{x}\right)\right]. \quad \text{This is incorrect!} \tag{3.6}$$

However, the graph of $y = \cos\left(\frac{1}{x}\right)$ found in Figure 1.19 suggests that $\cos\left(\frac{1}{x}\right)$ oscillates back and forth between -1 and 1. Further, the closer x gets to 0, the more rapid the oscillations become. You should compute some function values, as well, to convince yourself that $\lim_{x\to 0}\cos\left(\frac{1}{x}\right)$ does not exist. Equation (3.6) then does not hold and we're back to square one. Since none of our rules seem to apply here, the most reasonable step is to draw a graph and compute some function values in an effort to see what is going on. The graph of $y = x^2\cos\left(\frac{1}{x}\right)$ appears in Figure 1.20 and a table of function values is shown in the margin.

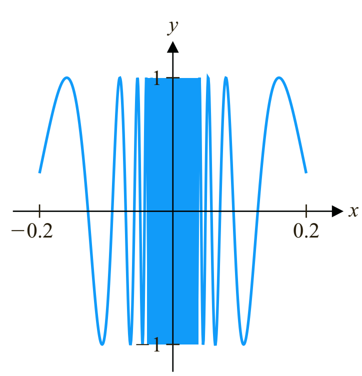

FIGURE 1.19
$y = \cos\left(\dfrac{1}{x}\right)$

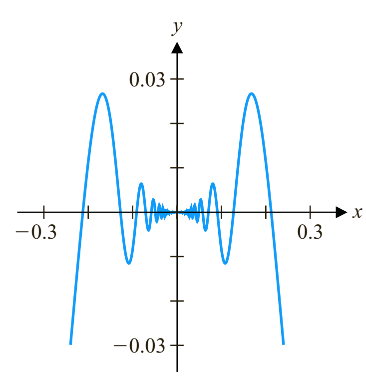

FIGURE 1.20
$y = x^2\cos\left(\dfrac{1}{x}\right)$

x	$x^2\cos(1/x)$
± 0.1	-0.008
± 0.01	8.6×10^{-5}
± 0.001	5.6×10^{-7}
± 0.0001	-9.5×10^{-9}
± 0.00001	-9.99×10^{-11}

The graph and the table of function values suggest the conjecture

$$\lim_{x\to 0}\left[x^2\cos\left(\frac{1}{x}\right)\right] = 0,$$

which we prove using the Squeeze Theorem. First, we need to find functions f and h such that

$$f(x) \le x^2\cos\left(\frac{1}{x}\right) \le h(x),$$

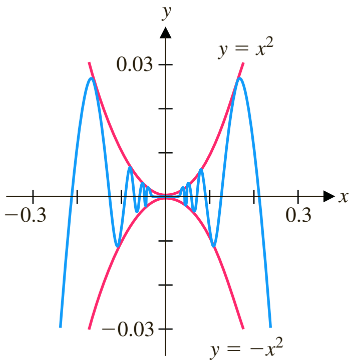

FIGURE 1.21
$y = x^2 \cos\left(\frac{1}{x}\right)$, $y = x^2$ and $y = -x^2$

for all $x \neq 0$ and where $\lim_{x\to 0} f(x) = \lim_{x\to 0} h(x) = 0$. Recall that

$$-1 \leq \cos\left(\frac{1}{x}\right) \leq 1, \tag{3.7}$$

for all $x \neq 0$. If we multiply (3.7) through by x^2 (notice that since $x^2 \geq 0$, this multiplication preserves the inequalities), we get

$$-x^2 \leq x^2 \cos\left(\frac{1}{x}\right) \leq x^2,$$

for all $x \neq 0$. We illustrate this inequality in Figure 1.21. Further,

$$\lim_{x\to 0}(-x^2) = 0 = \lim_{x\to 0} x^2.$$

So, from the Squeeze Theorem, it now follows that

$$\lim_{x\to 0} x^2 \cos\left(\frac{1}{x}\right) = 0,$$

also, as we had conjectured. ■

BEYOND FORMULAS

To resolve the limit in example 3.8, we could not apply the rules for limits contained in Theorem 3.1. So, we resorted to an indirect method of finding the limit. This tour de force of graphics plus calculation followed by analysis is sometimes referred to as the **Rule of Three.** (The Rule of Three presents a general strategy for attacking new problems. The basic idea is to look at problems graphically, numerically and analytically.) In the case of example 3.8, the first two elements of this "rule" (the graphics in Figure 1.20 and the accompanying table of function values) suggest a plausible conjecture, while the third element provides us with a careful mathematical verification of the conjecture. In what ways does this sound like the scientific method?

TODAY IN MATHEMATICS

Michael Freedman (1951–)
An American mathematician who first solved one of the most famous problems in mathematics, the four-dimensional Poincaré conjecture. A winner of the Fields Medal, the mathematical equivalent of the Nobel Prize, Freedman says, "Much of the power of mathematics comes from combining insights from seemingly different branches of the discipline. Mathematics is not so much a collection of different subjects as a way of thinking. As such, it may be applied to any branch of knowledge." Freedman finds mathematics to be an open field for research, saying that, "It isn't necessary to be an old hand in an area to make a contribution."

Functions are often defined by different expressions on different intervals. Such **piecewise-defined** functions are important and we illustrate such a function in example 3.9.

EXAMPLE 3.9 A Limit for a Piecewise-Defined Function

Evaluate $\lim_{x\to 0} f(x)$, where f is defined by

$$f(x) = \begin{cases} x^2 + 2\cos x + 1, & \text{for } x < 0 \\ e^x - 4, & \text{for } x \geq 0 \end{cases}.$$

Solution Since f is defined by different expressions for $x < 0$ and for $x \geq 0$, we must consider one-sided limits. We have

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (x^2 + 2\cos x + 1) = 2\cos 0 + 1 = 3,$$

by Theorem 3.4. Also, we have

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (e^x - 4) = e^0 - 4 = 1 - 4 = -3.$$

Since the one-sided limits are different, we have that $\lim_{x\to 0} f(x)$ does not exist. ■

We end this section with an example of the use of limits in computing velocity. In section 2.1, we see that for an object moving in a straight line, whose position at time t is given by the function $f(t)$, the instantaneous velocity of that object at time $t = 1$ (i.e., the velocity at the *instant* $t = 1$, as opposed to the average velocity over some period of time) is given by the limit

$$\lim_{h\to 0} \frac{f(1+h) - f(1)}{h}.$$

EXAMPLE 3.10 Evaluating a Limit Describing Velocity

Suppose that the position function for an object at time t (seconds) is given by

$$f(t) = t^2 + 2 \text{ (feet)},$$

find the instantaneous velocity of the object at time $t = 1$.

Solution Given what we have just learned about limits, this is now an easy problem to solve. We have

$$\lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{[(1+h)^2 + 2] - 3}{h}.$$

While we can't simply substitute $h = 0$ (why not?), we can write

$$\lim_{h\to 0} \frac{[(1+h)^2 + 2] - 3}{h} = \lim_{h\to 0} \frac{(1 + 2h + h^2) - 1}{h}$$

Expanding the squared term.

$$= \lim_{h\to 0} \frac{2h + h^2}{h} = \lim_{h\to 0} \frac{h(2+h)}{h}$$

$$= \lim_{h\to 0} \frac{2+h}{1} = 2.$$

Canceling factors of h.

So, the instantaneous velocity of this object at time $t = 1$ is 2 feet per second. ■

EXERCISES 1.3

WRITING EXERCISES

1. Given your knowledge of the graphs of polynomials, explain why equations (3.1) and (3.2) and Theorem 3.2 are obvious. Name five non-polynomial functions for which limits can be evaluated by substitution.
2. Suppose that you can draw the graph of $y = f(x)$ without lifting your pencil from your paper. Explain why $\lim_{x\to a} f(x) = f(a)$, for every value of a.
3. In one or two sentences, explain the Squeeze Theorem. Use a real-world analogy (e.g., having the functions represent the locations of three people as they walk) to indicate why it is true.
4. Given the graph in Figure 1.20 and the calculations that follow, it may be unclear why we insist on using the Squeeze Theorem before concluding that $\lim_{x\to 0}[x^2 \cos(1/x)]$ is indeed 0. Review section 1.2 to explain why we are being so fussy.

In exercises 1–34, evaluate the indicated limit, if it exists. Assume that $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

1. $\lim_{x\to 0}(x^2 - 3x + 1)$
2. $\lim_{x\to 2} \sqrt[3]{2x+1}$
3. $\lim_{x\to 0} \cos^{-1}(x^2)$
4. $\lim_{x\to 2} \frac{x-5}{x^2+4}$
5. $\lim_{x\to 3} \frac{x^2 - x - 6}{x-3}$
6. $\lim_{x\to 1} \frac{x^2 + x - 2}{x^2 - 3x + 2}$
7. $\lim_{x\to 2} \frac{x^2 - x - 2}{x^2 - 4}$
8. $\lim_{x\to 1} \frac{x^3 - 1}{x^2 + 2x - 3}$
9. $\lim_{x\to 0} \frac{\sin x}{\tan x}$
10. $\lim_{x\to 0} \frac{\tan x}{x}$
11. $\lim_{x\to 0} \frac{xe^{-2x+1}}{x^2 + x}$
12. $\lim_{x\to 0^+} x^2 \csc^2 x$

13. $\lim_{x\to 0} \dfrac{\sqrt{x+4}-2}{x}$

14. $\lim_{x\to 0} \dfrac{2x}{3-\sqrt{x+9}}$

15. $\lim_{x\to 1} \dfrac{x-1}{\sqrt{x}-1}$

16. $\lim_{x\to 4} \dfrac{\sqrt{x}-2}{x-4}$

17. $\lim_{x\to 1} \left(\dfrac{1}{x-1}-\dfrac{2}{x^2-1}\right)$

18. $\lim_{x\to 0} \left(\dfrac{2}{x}-\dfrac{2}{|x|}\right)$

19. $\lim_{x\to 0} \dfrac{1-e^{2x}}{1-e^x}$

20. $\lim_{x\to 0} \sin(e^{-1/x^2})$

21. $\lim_{x\to 0} \dfrac{\sin|x|}{x}$

22. $\lim_{x\to 0} \dfrac{\sin^2(x^2)}{x^4}$

23. $\lim_{x\to 2} f(x)$, where $f(x) = \begin{cases} 2x & \text{if } x<2 \\ x^2 & \text{if } x \ge 2 \end{cases}$

24. $\lim_{x\to -1} f(x)$, where $f(x) = \begin{cases} x^2+1 & \text{if } x<-1 \\ 3x+1 & \text{if } x \ge -1 \end{cases}$

25. $\lim_{x\to 0} f(x)$, where $f(x) = \begin{cases} x^2+2 & \text{if } x<-1 \\ 3x+1 & \text{if } x \ge -1 \end{cases}$

26. $\lim_{x\to 1} f(x)$, where $f(x) = \begin{cases} 2x & \text{if } x<2 \\ x^2 & \text{if } x \ge 2 \end{cases}$

27. $\lim_{x\to -1} f(x)$, where $f(x) = \begin{cases} 2x+1 & \text{if } x<-1 \\ 3 & \text{if } -1<x<1 \\ 2x+1 & \text{if } x>1 \end{cases}$

28. $\lim_{x\to 1} f(x)$, where $f(x) = \begin{cases} 2x+1 & \text{if } x<-1 \\ 3 & \text{if } -1<x<1 \\ 2x+1 & \text{if } x>1 \end{cases}$

29. $\lim_{h\to 0} \dfrac{(2+h)^2-4}{h}$

30. $\lim_{h\to 0} \dfrac{(1+h)^3-1}{h}$

31. $\lim_{h\to 0} \dfrac{h^2}{\sqrt{h^2+h+3}-\sqrt{h+3}}$

32. $\lim_{x\to 0} \dfrac{\sqrt{x^2+x+4}-2}{x^2+x}$

33. $\lim_{t\to -2} \dfrac{\frac{1}{2}+\frac{1}{t}}{2+t}$

34. $\lim_{x\to 0} \dfrac{\tan 2x}{5x}$

35. Use numerical and graphical evidence to conjecture the value of $\lim_{x\to 0} x^2 \sin(1/x)$. Use the Squeeze Theorem to prove that you are correct: identify the functions f and h, show graphically that $f(x) \le x^2 \sin(1/x) \le h(x)$ and justify $\lim_{x\to 0} f(x) = \lim_{x\to 0} h(x)$.

36. Why can't you use the Squeeze Theorem as in exercise 35 to prove that $\lim_{x\to 0} x^2 \sec(1/x) = 0$? Explore this limit graphically.

37. Use the Squeeze Theorem to prove that $\lim_{x\to 0^+} [\sqrt{x}\cos^2(1/x)] = 0$. Identify the functions f and h, show graphically that $f(x) \le \sqrt{x}\cos^2(1/x) \le h(x)$ for all $x>0$, and justify $\lim_{x\to 0^+} f(x) = 0$ and $\lim_{x\to 0^+} h(x) = 0$.

38. Suppose that $f(x)$ is bounded: that is, there exists a constant M such that $|f(x)| \le M$ for all x. Use the Squeeze Theorem to prove that $\lim_{x\to 0} x^2 f(x) = 0$.

In exercises 39–42, either find the limit or explain why it does not exist.

39. $\lim_{x\to 4^+} \sqrt{16-x^2}$

40. $\lim_{x\to 4^-} \sqrt{16-x^2}$

41. $\lim_{x\to -2^-} \sqrt{x^2+3x+2}$

42. $\lim_{x\to -2^+} \sqrt{x^2+3x+2}$

43. Given that $\lim_{x\to 0^+} \dfrac{1-\cos x}{x^2} = \dfrac{1}{2}$, quickly evaluate $\lim_{x\to 0^+} \dfrac{\sqrt{1-\cos x}}{x}$.

44. Given that $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$, quickly evaluate $\lim_{x\to 0} \dfrac{1-\cos^2 x}{x^2}$.

45. Suppose $f(x) = \begin{cases} g(x) & \text{if } x<a \\ h(x) & \text{if } x>a \end{cases}$ for polynomials $g(x)$ and $h(x)$. Explain why $\lim_{x\to a^-} f(x) = g(a)$ and determine $\lim_{x\to a^+} f(x)$.

46. Explain how to determine $\lim_{x\to a} f(x)$ if g and h are polynomials and $f(x) = \begin{cases} g(x) & \text{if } x<a \\ c & \text{if } x=a \\ h(x) & \text{if } x>a \end{cases}$.

47. Evaluate each limit and justify each step by citing the appropriate theorem or equation.

(a) $\lim_{x\to 2} (x^2-3x+1)$ (b) $\lim_{x\to 0} \dfrac{x-2}{x^2+1}$

48. Evaluate each limit and justify each step by citing the appropriate theorem or equation.

(a) $\lim_{x\to -1} [(x+1)\sin x]$ (b) $\lim_{x\to 1} \dfrac{xe^x}{\tan x}$

In exercises 49–52, use the given position function $f(t)$ to find the velocity at time $t = a$.

49. $f(t) = t^2+2,\ a=2$

50. $f(t) = t^2+2,\ a=0$

51. $f(t) = t^3,\ a=0$

52. $f(t) = t^3,\ a=1$

53. In Chapter 2, the slope of the tangent line to the curve $y = \sqrt{x}$ at $x=1$ is defined by $m = \lim_{h\to 0} \dfrac{\sqrt{1+h}-1}{h}$. Compute the slope m. Graph $y=\sqrt{x}$ and the line with slope m through the point $(1, 1)$.

54. In Chapter 2, an alternative form for the limit in exercise 53 is given by $\lim_{x\to 1} \dfrac{\sqrt{x}-1}{x-1}$. Compute this limit.

In exercises 55–62, use numerical evidence to conjecture the value of the limit if it exists. Check your answer with your Computer Algebra System (CAS). If you disagree, which one of you is correct?

55. $\lim_{x\to 0^+} (1+x)^{1/x}$

56. $\lim_{x\to 0^+} e^{1/x}$

57. $\lim_{x\to 0^+} x^{-x^2}$

58. $\lim_{x\to 0^+} x^{\ln x}$

59. $\lim_{x\to 0} \sin\dfrac{1}{x}$

60. $\lim_{x\to 0} e^{1/x}$

61. $\lim_{x\to 0} \tan^{-1}\dfrac{1}{x}$

62. $\lim_{x\to 0} \ln\left|\dfrac{1}{x}\right|$

In exercises 63–66, use $\lim_{x\to a} f(x) = 2$, $\lim_{x\to a} g(x) = -3$ and $\lim_{x\to a} h(x) = 0$ to determine the limit, if possible.

63. $\lim_{x\to a} [2f(x)-3g(x)]$

64. $\lim_{x\to a} [3f(x)g(x)]$

65. $\lim_{x\to a} \left[\dfrac{f(x)+g(x)}{h(x)}\right]$

66. $\lim_{x\to a} \left[\dfrac{3f(x)+2g(x)}{h(x)}\right]$

67. Assume that $\lim_{x\to a} f(x) = L$. Use Theorem 3.1 to prove that $\lim_{x\to a}[f(x)]^3 = L^3$. Also, show that $\lim_{x\to a}[f(x)]^4 = L^4$.

68. How did you work exercise 67? You probably used Theorem 3.1 to work from $\lim_{x\to a}[f(x)]^2 = L^2$ to $\lim_{x\to a}[f(x)]^3 = L^3$, and then used $\lim_{x\to a}[f(x)]^3 = L^3$ to get $\lim_{x\to a}[f(x)]^4 = L^4$. Going one step at a time, we should be able to reach $\lim_{x\to a}[f(x)]^n = L^n$ for any positive integer n. This is the idea of **mathematical induction.** Formally, we need to show the result is true for a specific value of $n = n_0$ [we show $n_0 = 2$ in the text], then assume the result is true for a general $n = k \ge n_0$. If we show that we can get from the result being true for $n = k$ to the result being true for $n = k + 1$, we have proved that the result is true for any positive integer n. In one sentence, explain why this is true. Use this technique to prove that $\lim_{x\to a}[f(x)]^n = L^n$ for any positive integer n.

69. Find all the errors in the following incorrect string of equalities:

$$\lim_{x\to 0}\frac{1}{x} = \lim_{x\to 0}\frac{x}{x^2} = \lim_{x\to 0} x \lim_{x\to 0}\frac{1}{x^2} = 0 \cdot ? = 0.$$

70. Find all the errors in the following incorrect string of equalities:

$$\lim_{x\to 0}\frac{\sin 2x}{x} = \frac{0}{0} = 1.$$

71. Give an example of functions f and g such that $\lim_{x\to 0}[f(x) + g(x)]$ exists but $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} g(x)$ do not exist.

72. Give an example of functions f and g such that $\lim_{x\to 0}[f(x)\cdot g(x)]$ exists but at least one of $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} g(x)$ does not exist.

73. If $\lim_{x\to a} f(x)$ exists and $\lim_{x\to a} g(x)$ does not exist, is it always true that $\lim_{x\to a}[f(x) + g(x)]$ does not exist? Explain.

74. Is the following true or false? If $\lim_{x\to 0} f(x)$ does not exist, then $\lim_{x\to 0}\frac{1}{f(x)}$ does not exist. Explain.

75. Suppose a state's income tax code states the tax liability on x dollars of taxable income is given by

$$T(x) = \begin{cases} 0.14x & \text{if } 0 \le x < 10{,}000 \\ 1500 + 0.21x & \text{if } 10{,}000 \le x \end{cases}.$$

Compute $\lim_{x\to 0^+} T(x)$; why is this good? Compute $\lim_{x\to 10{,}000} T(x)$; why is this bad?

76. Suppose a state's income tax code states that tax liability is 12% on the first \$20,000 of taxable earnings and 16% on the remainder. Find constants a and b for the tax function $T(x) = \begin{cases} a + 0.12x & \text{if } x \le 20{,}000 \\ b + 0.16(x - 20{,}000) & \text{if } x > 20{,}000 \end{cases}$ such that $\lim_{x\to 0^+} T(x) = 0$ and $\lim_{x\to 20{,}000} T(x)$ exists. Why is it important for these limits to exist?

77. The **greatest integer function** is denoted by $f(x) = [x]$ and equals the greatest integer that is less than or equal to x. Thus, $[2.3] = 2$, $[-1.2] = -2$ and $[3] = 3$. In spite of this last fact, show that $\lim_{x\to 3}[x]$ does not exist.

78. Investigate the existence of (a) $\lim_{x\to 1}[x]$, (b) $\lim_{x\to 1.5}[x]$, (c) $\lim_{x\to 1.5}[2x]$, and (d) $\lim_{x\to 1}(x - [x])$.

EXPLORATORY EXERCISES

1. The value $x = 0$ is called a **zero of multiplicity** $\boldsymbol{n}$ $(n \ge 1)$ for the function f if $\lim_{x\to 0}\frac{f(x)}{x^n}$ exists and is nonzero but $\lim_{x\to 0}\frac{f(x)}{x^{n-1}} = 0$. Show that $x = 0$ is a zero of multiplicity 2 for x^2, $x = 0$ is a zero of multiplicity 3 for x^3 and $x = 0$ is a zero of multiplicity 4 for x^4. For polynomials, what does multiplicity describe? The reason the definition is not as straightforward as we might like is so that it can apply to non-polynomial functions, as well. Find the multiplicity of $x = 0$ for $f(x) = \sin x$; $f(x) = x\sin x$; $f(x) = \sin x^2$. If you know that $x = 0$ is a zero of multiplicity m for $f(x)$ and multiplicity n for $g(x)$, what can you say about the multiplicity of $x = 0$ for $f(x) + g(x)$? $f(x)\cdot g(x)$? $f(g(x))$?

2. We have conjectured that $\lim_{x\to 0}\frac{\sin x}{x} = 1$. Using graphical and numerical evidence, conjecture the value of $\lim_{x\to 0}\frac{\sin 2x}{x}$, $\lim_{x\to 0}\frac{\sin 3x}{x}$, $\lim_{x\to 0}\frac{\sin \pi x}{x}$ and $\lim_{x\to 0}\frac{\sin x/2}{x}$. In general, conjecture the value of $\lim_{x\to 0}\frac{\sin cx}{x}$ for any constant c. Given that $\lim_{x\to 0}\frac{\sin cx}{cx} = 1$ for any constant $c \ne 0$, prove that your conjecture is correct.

1.4 CONTINUITY AND ITS CONSEQUENCES

When you describe something as *continuous,* just what do you have in mind? For example, if told that a machine has been in *continuous* operation for the past 60 hours, most of us would interpret this to mean that the machine has been in operation *all* of that time, without